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FINITE ELEMENT ANALYSIS OF INCOMPRESSIBLE MATERIAL BY RESIDUAL ENERGY BALANCING

ISAAC FRIED†

Boston University, Department of Mathematics,
Boston, Massachusetts 02215, U.S.A.

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Abstract—Incompressibility is gradually introduced into the finite elements with the mesh refinement in such a way as to balance it with the residual discretization energy in order to ensure fastest convergence to the incompressible solution with best conditioned stiffness matrix to minimize round-off errors in the computations.

INTRODUCTION

In the finite element analysis of elastic matter, as Poisson's ratio nears one half or as the material approaches incompressibility two things might happen: the approximation quality of the elements can be entirely lost or the condition of the stiffness matrix may deteriorate indefinitely[1]. Assuming beforehand the material incompressible gives rise to a new independent variable—the pressure and the appearance of an additional equation for zero volume change. Variational principles, from which incompressible finite elements can be formed, including the pressure as a Lagrange multiplier have been brought forth[2] but are no more minimal.

Yet, even if the analytical model assumes the material incompressible, computationally it is rarely so owing to the discretization. It is reasonable, therefore, to attempt to introduce the incompressibility only gradually as the mesh is refined in such a manner as to balance it with the discretization error and avoid excessive ill-conditioned stiffness matrix. The purpose of this paper is to demonstrate the practical feasibility of such a scheme.

Successful employment of this residual energy balancing technique to the finite element modeling of thin plates and shells is described in[3–5].

COMPRESSIBILITY ERROR

In this section we derive the energy error incurred in replacing the incompressible material by a compressible one.

The equations of equilibrium of an elastic solid enclosed in D with surface $S = S_1 + S_2$ are

$$\nabla^2 u_i + \frac{m}{m-2}\frac{\partial e}{\partial x_i} + F_i/G = 0 \qquad i = 1, 2, 3, \text{ in } D \tag{1}$$

with

$$e = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \text{Divergence } U \tag{2}$$

† Assistant Professor.

and with the prescribed displacements and surface traction boundary conditions

$$U = (u_1^*, u_2^*, u_3^*) \quad \text{on} \quad S_1 \tag{3}$$

$$T = (T_1^*, T_2^*, T_3^*) \quad \text{on} \quad S_2 \tag{4}$$

where $m = 1/\nu$ is the inverse of Poisson's ratio, G the shear modulus, e the volume change. $U = (u_1, u_2, u_3)$ the displacement vector in the Cartesian system (x_1, x_2, x_3), $F = (F_1, F_2, F_3)$ the body forces, and $T = (T_1, T_2, T_3)$ the surface tractions. The strain tensor ε_{ij} is given in terms of U by

$$\varepsilon_{ij} = \frac{\partial u_i}{\partial x_i}, \quad \varepsilon_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \qquad i, j = 1, 2, 3. \tag{5}$$

Hook's law which relates the stress σ_{ij} to the strain ε_{ij} can be written for the isotropic solid as

$$\sigma_{ii} = 2G\left(\varepsilon_{ii} + \frac{e}{m-2}\right), \qquad \sigma_{ij} = G\varepsilon_{ij} \qquad i, j = 1, 2, 3 \tag{6}$$

and on any surface with unit normal vector $n = (n_1, n_2, n_3)$ the traction is

$$T_i = \sigma_{i1} n_1 + \sigma_{i2} n_2 + \sigma_{i3} n_3. \tag{7}$$

The total potential energy $\pi(U)$ of the boundary value problem in equations (1)–(3) and (4) we write as

$$\pi(U) = E(U) - \int_D (F_1 u_1 + F_2 u_2 + F_3 u_3)\, dv - \int_{S_2} (T_1^* u_1 + T_2^* u_2 + T_3^* u_3)\, ds \tag{8}$$

where the elastic energy $E(U)$ is explicitly given by

$$E(U) = \int_D \left\{ \frac{1}{3}\frac{m+1}{m-2} e^2 + \frac{1}{2}[(\varepsilon_{11} - \varepsilon_{22})^2 + (\varepsilon_{22} - \varepsilon_{33})^2 + (\varepsilon_{33} - \varepsilon_{11})^2] + \tfrac{1}{2}(\varepsilon_{12}^2 + \varepsilon_{23}^2 + \varepsilon_{31}^2) \right\} dv \tag{9}$$

with G set equal 1. The principle of minimum potential energy states that if U is the elastic solution to equations (1)–(4) then

$$\pi(U) = \min_{\tilde{U}} \pi(\tilde{U}) \tag{10}$$

minimization being carried out of the displacement trial field $\tilde{U}$ which satisfy equation (3) and for which $E(\tilde{U}) < \infty$.

In the case of an incompressible material with $m = 2$ (or $\nu = 1/2$) the pressure p

$$p = \tfrac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \tag{11}$$

becomes independent and the direct stress strain relations become with unit G

$$\sigma'_{ii} = 2\frac{\partial u_i'}{\partial x_i} + p \qquad i = 1, 2, 3 \tag{12}$$

while the equations of equilibrium become

$$\nabla^2 u_i' + \frac{\partial p}{\partial x_i} + F_i = 0 \tag{13}$$

in which ( )′ refers to the incompressible state.

Integration by parts with the use of equations (12) and (13) proves that the same displacement function that minimizes $\pi(U)$ in equation (8) minimizes also

$$\begin{aligned}\psi(U) = \int_D \Big\{ &\tfrac{1}{3}[(\varepsilon_{11} - \varepsilon_{22} - \varepsilon_{11}' + \varepsilon_{22}')^2 + (\varepsilon_{22} - \varepsilon_{33} - \varepsilon_{22}' + \varepsilon_{33}')^2 \\ &+ (\varepsilon_{33} - \varepsilon_{11} - \varepsilon_{33}' + \varepsilon_{11}')^2] + \tfrac{1}{2}[(\varepsilon_{12} - \varepsilon_{12}')^2 + (\varepsilon_{23} - \varepsilon_{23}')^2 \\ &+ (\varepsilon_{31} - \varepsilon_{31}')^2] + \frac{1}{2K}(Ke - p)^2 \Big\} \, dv\end{aligned} \tag{14}$$

which can be concisely written as

$$\psi(U) = \|U - U'\|_1'^2 + \tfrac{1}{2}K \left\| e - \frac{1}{K} p \right\|_0^2 \tag{15}$$

and in which

$$K = \frac{2}{3}\frac{m+1}{m-2}. \tag{16}$$

Since U minimizes $\psi(U)$

$$\psi(U) \leq \psi(U') \tag{17}$$

and consequently

$$\|U - U'\|_1'^2 + \tfrac{1}{2}K \left\| e - \frac{1}{K} p \right\|_0^2 \leq \frac{1}{2K} \|p\|_0^2 \tag{18}$$

leading to

$$\left\| e - \frac{p}{K} \right\|_0 \leq \left(\frac{1}{K^2} \|p\|_0^2 - \frac{2}{K} \|U - U'\|_1'^2 \right)^{1/2} \tag{19}$$

which with

$$\left\| e - \frac{1}{K} p \right\|_0 \geq \|e\|_0 - \frac{1}{K} \|p\|_0 \tag{20}$$

becomes

$$\|U - U'\|_1'^2 + \tfrac{1}{2}K \|e\|_0^2 \leq \frac{2}{K} \|p\|_0^2 \tag{21}$$

or equivalently

$$E(U - U') \leq \frac{2}{K} \int_D p^2 \, dv \tag{22}$$

and this is what we were seeking.

To prove that the bound in equation (22) is optimal and that in fact $E(U - U')$ varies linearly with $1/K$, we only need find an example where equality in equation (22) actually occurs. An example with an accessible analytical solution is that of a hollow sphere[6] holding an internal unit pressure. In the incompressible case the radial displacement u' and the pressure are given by

$$u' = \frac{A}{r^2} \qquad A = \frac{a^3 b^3}{4(b^3 - a^3)}, \qquad p = \frac{a^3}{b^3 - a^3} \tag{23}$$

in which r is the radial coordinate and a and b the inner and outer radii, respectively. In the compressible case

$$u = \frac{A}{r^2} + \frac{m - 2}{2(m + 1)} pr \tag{24}$$

from which we find that

$$E(u - u') = \tfrac{1}{2} p^2 / K \tag{25}$$

assuming unit volume.

CONDITION OF STIFFNESS MATRIX

According to[1] the spectral condition number $C_2(K)$ of the global stiffness matrix formed from the elastic energy expression in equation (9) is bounded by

$$c_2 \frac{N_{es}^2}{\mu_1(m - 2)} \leq C_2(K) \leq c_1 \frac{N_{es}^2}{\lambda_1(m - 2)} \tag{26}$$

in which N_{es} is the number of elements per side in the mesh, c_1 and c_2 numerical constants, λ_1 the exact lowest eigenvalue of the solid and μ_1 the approximate finite element eigenvalue. Once the approximation is good enough for μ_1 to be replaced by λ_1 the condition number becomes of the form

$$C_2(K) = c_3 \frac{N_{es}^2}{\lambda_1(m - 2)}. \tag{27}$$

Our ability to obtain a good approximation for λ_1 when K in equation (16) is large with a reasonable number (say seven) of finite elements may well depend of K because of the term

$$\tfrac{1}{2} K \int_D e^2 \, dv \tag{28}$$

in the elastic energy expression and we will pay close attention to this crucial approximation question in the following sections. It will be shown that indeed some finite elements are entirely inadequate to cope with $K \to \infty$ while others furnish accurate results regardless of $K \to \infty$ permitting the interchange of λ_1 and μ_1 in equation (26) for discretization with a moderate number of finite elements.

In any event care must be exercised in assigning a value for $m = 1/\nu$ in order to approximate the incompressible state. Lowering ν too much away from 1/2 might introduce a large compressibility error compared with the accuracy which can be obtained with the mesh of finite elements. On the other hand taking ν too close to 1/2 might either ruin the discretization accuracy or produce excessively ill-conditioned stiffness matrix. Poisson's ratio $\nu = 1/m$ ought

to be increased gradually with the mesh refinement to balance the compressibility error with the best discretization accuracy and in such a way as to keep $C_2(K)$ as low as possible.

Exploring ways to accomplish this is the subject of the following sections.

RESIDUAL ENERGY BALANCING

We introduce the residual energy balancing technique by way of carefully examining the problem of a radially pulsating hollow sphere free of surface and body forces. Presently

$$e = u_r + 2u/r \tag{29}$$

and to form the element stiffness and mass matrices we need the elastic and kinetic energies E and K which we write here as

$$E = G\int_a^b \left[\frac{1}{m-2}e^2 + \left(u_r^2 + 2\frac{u^2}{r^2}\right)\right] r^2\,\mathrm{d}r \tag{30}$$

and

$$K = \frac{1}{2}\int_a^b u^2 r^2\,\mathrm{d}r. \tag{31}$$

The element stiffness matrix k derived from E in equation (30) we prefer to write in the form

$$k = zk_1 + Gk_2, \qquad z = G/(m-2). \tag{32}$$

Assembling the element stiffness matrices k into the global stiffness matrix K and the element mass matrices m into the global mass matrix M produces the algebraic eigenproblem

$$(zK_1 + GK_2)x = \lambda Mx \tag{33}$$

and the lowest λ is given by

$$\lambda = \min_x \left(z\frac{x^TK_1x}{x^TMx} + G\frac{x^TK_2x}{x^TMx}\right). \tag{34}$$

The term zx^TK_1x/x^TMx is the compressibility energy and should vanish as $z \to \infty$, but we have to distinguish here between two distinct cases: the one with a singular K_1 and the other with a non singular K_1.

Suppose first that discretization is achieved by assuming a polynomial approximation of degree p and in particular let $p = 1$. For this, the (lumped[7]) element matrix m and the element matrix k_1 for an element situated between $r = r_1$ and $r = r_2$ become

$$k_1 = \frac{h}{4}\begin{pmatrix}\alpha^2 + 3/4 & \alpha\beta - 3/4\\ \alpha\beta - 3/4 & \beta^2 + 3/4\end{pmatrix}, \qquad m = \frac{h}{4}\begin{pmatrix}r_1^2 & \\ & r_2^2\end{pmatrix} \tag{35}$$

in which $r_a = (r_1 + r_2)/2$, $h = r_2 - r_1$, $\alpha = 1 - r_a/h$ and $\beta = 1 + r_a/h$. The element stiffness matrix k_1 is non singular and since[8]

$$\min_x \frac{x^TK_1x}{x^Tx} \geq \min_y \frac{y^Tk_1y}{y^Ty} \tag{36}$$

so will K_1 be and we have that

$$\min \frac{x^TK_1x}{x^TMx} = 0(h^2). \tag{37}$$

This means that incompressibility is asymptotically achieved as $h \to 0$ but for any given h accuracy is ruined by increasing z indefinitely since then $zx^T K_1 x / x^T M x$ will grow without bound making $\lambda = 0$ or the element perfectly rigid.

In boundary value problems the situation is the same. There the energy error $E(u - \hat{u})$ in the finite element solution $\hat{u}$ (this is also the error in eigenvalues in eigenproblems[9, 10]) is

$$E(u - \hat{u}) = z \int_a^b (e - \hat{e})^2 r^2 \, dr + \int_a^b \left[(u_r - \hat{u}_r)^2 + \frac{2}{r^2} (u - \hat{u})^2 \right] r^2 \, dr \tag{38}$$

and if $\hat{e}$ can not assume any constant value unless $\hat{u} = 0$ then as $z \to \infty$, $e \to 0$, $\hat{e} \to 0$ but also $\hat{u} \to 0$ making the element ever more rigid.

This situation is entirely analogous to that in thin inextensional curved beams where if the extensional strain can not assume exactly the rigid body mode (i.e. assume any constant value) then reducing the thickness makes the extensional energy vanish only with vanishing displacements and the element becomes excessively rigid.

In view of equation (37) the discretization error can be balanced with the compressibility error by choosing, for first order elements, $z = 0(h^{-1})$ since then the compressibility error derived before is $0(h)$ and so will be $zx^T K_1 x / x^T M x$. But it is rather a poor approximation since with linear elements we expect quadratic accuracy for the eigenvalues.

In boundary value problems the situation is again the same. The second integral in equation (38) can be $0(h^2)$ but the first only $0(h)$. Numerical experiments confirm too that no more than $0(h)$ accuracy is obtainable in this case.

To avoid the absolute rigidity of the element with $z = \infty$ we need singularize K_1 and can achieve this by choosing the shape functions to enable $\hat{e}$ to assume any constant value (in curved beams or shells this is analogous to the introduction of exact rigid body modes). Thus we start by assuming a polynomial assumption for $\hat{e}$ (as in[5]) and integrating. For

$$\hat{e} = \hat{u}_r + 2 \frac{\hat{u}}{r} = a_0 \tag{39}$$

this yields

$$\hat{u} = \frac{b_0}{r^2} + \tfrac{1}{3} a_0 r \tag{40}$$

and with these shape functions both k_1 and K_1 become singular. Now minimization in equation (34) with $z \to \infty$ will seek the mode corresponding to the zero eigenvalue of K_1 and $zx^T K_1 x / x^T M x$ will remain finite for any z. Since presently $zx^T K_1 / x^T M x$ can become arbitrarily small for any z we expect to find a mode that will yield simultaneously $0(h^2)$ accuracy for the second integral in equation (38) and $0(h^2)$ also for the first integral in equation (38). Another technically interesting possibility to reduce the rank of K_1, or at least hasten the diminishing of its lower spectrum, is by assuming a polynomial expressing for $\hat{u}$ and using numerical integration. If $\hat{u}$ is assumed a polynomial of degree p then following the rule of[11, 12] and integrating each energy term by p Gauss points will singularize k_1 and consequently K_1. For instance, with $p = 1$, analytic integration produce the k_1 (2×2) matrix of rank 2, but a one Gauss point integration reduces this to rank 1.

The beneficial effects of numerical integration in the finite element analysis of thick plates and shells were earlier discovered and described in[13–15].

NUMERICAL EXPERIMENTS

The purpose of these experiments is to verify, at least for the model problem of a freely vibrating hollow sphere made of incompressible material, that by approximating the displacement by a polynomial of degree p and integrating (in one dimension) the energy terms with p Gauss points the full energy rate of convergence $0(h^{2p})$ can be obtained by taking

$$z = cN_e^{-2p} \tag{41}$$

where c is a positive constant and N_e the number of elements (in multidimensional problems N_e should be replaced by N_{es}, the number of elements per side) A reasonable value for c keeping $C_2(K)$ as low as possible will also emerge from these experiments.

The fundamental eigenvalue (square of frequency) λ of the radially vibrating sphere is given by

$$\lambda = 4G\frac{a^2 + ab + b^2}{a^2b^2} \tag{42}$$

which for $G = \frac{1}{3}$ (unit elastic modulus), $a = \frac{1}{2}$ and $b = 1$ becomes $\lambda = \frac{28}{3} = 9{\cdot}333$.

First, the sphere is discretized with first order ($p = 1$), two-nodal-point elements and the stiffness matrix created with one Gauss point integration In accordance with equation (41) z is taken in the form $z = cN_e^2$ and the calculated eigenvalue λ is shown in Fig. 1 as a function of the number of elements in the mesh and for different values of c. The variation of λ with c is also shown in Fig. 2 for $N_e = 15$ and indeed between 2 and 50 the accuracy of the eigenvalue depends only weakly on c A reasonable value for c is seen from Fig. 2 to be 5. Figure 3 shows the reduction of the relative error $\delta\lambda/\lambda$ in λ vs N_e on a logarithmic scale to show that the optimal rate of convergence $0(N_e^{-2})$ is actually achieved. For the choice $z = cN_e^2$ it is found that $C_2(K) = 3cN_e^4$ (Fig. 4(a) shows $C_2(K)$ vs N_e for $c = 5$ on a logarithmic scale) and hence even though overestimating c has little effect on the accuracy, it raises the condition number of K and should be kept down. Actually to obtain three decimals accuracy in λ with linear ($p = 1$) elements and with $c = 5$ about 10 elements are required with $C_2(K) = 1{\cdot}5\ 10^5$, which is well within the capabilities of modern computers.

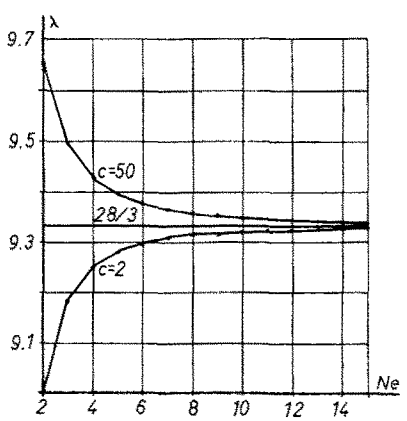

Fig. 1. Convergence of computed fundamental eigenvalue λ of a radially pulsating incompressible hollow sphere discretized with first order ($P = 1$) elements integrated with one Gauss point as a function of the number of elements in the mesh and for different values of c in $z = cN_e^2$.

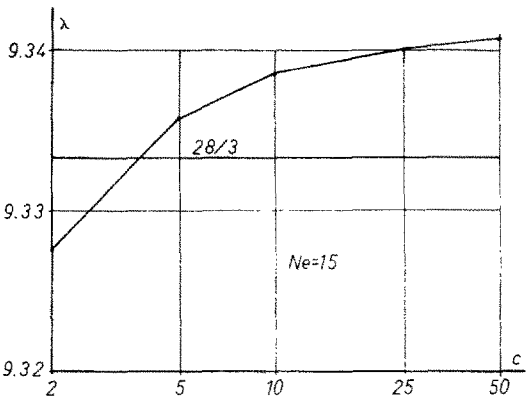

Fig. 2. Variation of λ from Fig. 1 with c for $N_e = 15$.

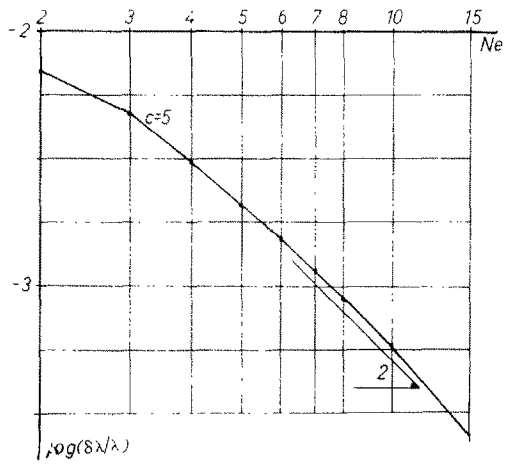

Fig. 3. Convergence of λ from Fig. 1 for $c = 5$ vs N_e. Logarithmic scale demonstrates the attainment of the optimal rate of convergence $0(N_e^{-2})$.

Increasing the order of the element must be accompanied by an increase in the rate of growth of z and consequently of $C_2(K)$. In general with a polynomial of degree p (and p Gauss integration points) $z = cN_e^{2p}$ and $C_2(K) = 0(N_e^{2p+2})$ which is unlike the compressible case in which the rate of growth of $C_2(K)$ does not depend on p. Nevertheless with higher order elements the error decreases like $0(N_e^{-2p})$ and fewer elements are needed for a given accuracy. Hence in spite of the fact that $C_2(K)$ grows faster with higher order elements, less elements are needed for the same accuracy and the value of $C_2(K)$ might still be lower than that for the low order elements. The following numerical example shows that this is actually, the case. It consists of computing λ with a quadratic ($p = 2$), three-nodal-point element, a 2 point Gauss integration and $z = cN_e^4$. Figure 5 shows the rate of convergence of λ for different values of c and this is seen indeed to be $0(N_e^{-4})$. A good choice for c is 10, but again overestimating it changes the discretization accuracy only slightly. The condition number grows now with N_e^5 as seen from Fig. 4 (b) (drawn for $c = 10$). Yet per accuracy the

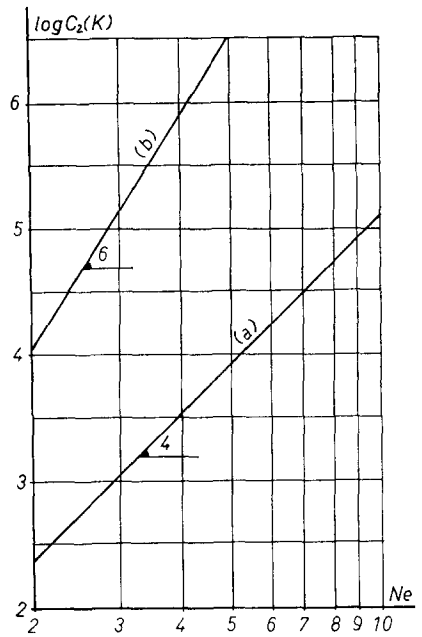

Fig. 4. Spectral condition number $C_2(K)$ of global matrix K for hollow sphere discretized with (a) linear elements integrated with one Gauss point and with $z = 5N_e^2$ and (b) quadratic elements integrated with two Gauss points and with $z = 10N_e^4$. Theoretical rates of growth $0(N_e^4)$ and $0(N_e^6)$ are verified.

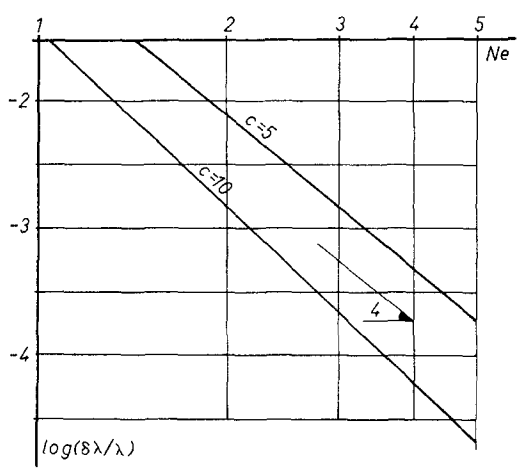

Fig. 5. Convergence of λ with quadratic ($p = 2$) elements. Optimal rate of convergence $0(N_e^{-4})$ is obtained with $z = cN_e^4$.

higher element leads to a better conditioned global stiffness matrix. To obtain a 3 decimals accuracy in λ only 3 quadratic elements are needed (they are hence also more economical computationally) and $C_2(K) = 2 \times 10^4$, less than that with linear ($p = 1$) elements.

REFERENCES

1. I. Fried, Influence of Poisson's ratio on the condition of the finite element stiffness matrix, *Int. J. Solids Struct.* **9**, 223–329 (1973).
2. R. L. Taylor, K. S. Pister and L. H. Herrmann, On a variational theorem for incompressible and nearly incompressible orthotropic elasticity, *Int. J. Solids Struct.* **4**, 875–883 (1968).
3. I. Fried, Shear in C^0 and C^1 bending finite elements, *Int. J. Solids Struct.* **9**, 449–460 (1973).
4. I. Fried and Shok Keng Yang, Triangular, nine-degrees-of-freedom, C^0 plate bending element of quadratic accuracy, *Quart. Appl. Math.* 303–312, (October 1973).
5. I. Fried, Shape functions and the accuracy of arch finite elements, *AIAA J.* **11** (3), 287–291 (1973).
6. A. E. H. Love, *The Mathematical Theory of Elasticity*, 4th. Edn., Section 98. Dover, New York(1944).
7. I. Fried, Lumping the finite element mass matrix with no accuracy loss. To be published.
8. I. Fried, Bounds on the spectral and maximum norms of the finite element stiffness, flexibility and mass matrices, *Int. J. Solids Struct.* **9**, 1013–1034 (1973).
9. I. Fried, Accuracy of finite element eigenproblems, *J. Sound Vibr.* **18** (2), 289–295 (1971).
10. I. Fried, Boundary and interior approximation errors in the finite element method, *J. App. Mech.* **40** (4), 1113–1117 (1973).
11. I. Fried, Accuracy and condition of curved (isoparametric) finite elements, *J. Sound Vibr.* **31** (3), 345–355 (1973).
12. I. Fried, Numerical integration in the finite element method, *Computers & Structures*, to be published.
13. W. P. Doherty, E. L. Wilson and R. L. Taylor, Stress analysis of axi-symmetric solids using higher order quadrilateral finite elements. *Structural Engineering Laboratory Report* No. SESM 69-3, University of California, Berkeley (1969).
14. O. C. Zienkiewicz, R. L. Taylor and J. M. Too, Reduced integration technique in general analysis of plates and shells, *Int. J. Num. Meth. Engng* **3**, 275–290 (1971).
15. S. F. Pawsey and R. W. Clough, Improved numerical integration of thick shell finite elements, *Int. J. Num. Meth. Engng* **3**, 575–586 (1971).